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A direct method for constructing periodic boundary-layer solutions

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An efficient method is proposed for solving steady problems involving the interaction between a boundary layer and an external stream with periodic boundary conditions, including reverse-flow regions. When non-uniqueness exists, the method can also be used to find additional solutions. The algorithm is described, and examples of such solutions are given.

Keywords: separation; method; interaction; boundary; periodic solution; non-uniqueness

1. Introduction

The interaction between a boundary layer and an inviscid external stream is one of the central problems in the theory of high-Reynolds-number separated flows. Even though the interaction can occur in different situations, the problem may be posed in a quite universal form. For two-dimensional flow, the problem is stated as follows (Sychev *et al.* 1998). The flow in the viscous sublayer is described by Prandtl's equations,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial Y} = -\frac{\mathrm{d}P}{\mathrm{d}x} + \frac{\partial^2 u}{\partial Y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial Y}, \tag{1.1}$$

supplemented with the boundary conditions,

$$u = v = 0; \qquad Y = 0, \qquad -\infty < x < \infty, \tag{1.2}$$

$$u \to \tau Y \quad \text{as} \quad Y \to \infty, \qquad -\infty < x < \infty, \qquad \tau = \text{const.} > 0. \tag{1.3}$$

In the standard non-periodic case, the initial velocity profile is typically given in the following asymptotic form:

$$u \to \tau Y \quad \text{as} \quad x \to -\infty, \qquad 0 \leqslant Y < \infty. \tag{1.4}$$

However, when the flow is periodic with respect to the streamwise coordinate x, this condition is replaced by the condition that the cross-sectional velocity profile at x is identical with that at $x + a$, namely,

$$u(Y, x) = u(Y, x + a), \qquad 0 \leqslant Y < \infty. \tag{1.5}$$

where the period a of the solution is a parameter of the problem. The initial pressure distribution along the interaction region is not known and must be determined by an interaction condition, i.e. a relation between pressure, displacement induced by the viscous sublayer and the geometry of the body placed in the stream. The form of the relation depends on the specific nature of the interaction and may involve some additional parameters, such as the flow rate or pressure drop. In the general case, the interaction condition can be written as

$$F(A(x), f(x), P(x)) = 0, \quad u \to \tau Y + A(x) + f(x) \quad \text{as} \quad Y \to \infty, \quad 0 \leqslant x \leqslant a.$$

Here, the following non-dimensional variables are used: x, Y is an orthogonal coordinate system with $Y = 0$ being along the body surface, u and v are the projections of the velocity vector onto the coordinate axes, P is the pressure disturbance, f describes the geometry of the body, A' is the slope of the displacement thickness, and F is some operator. The exact general solution to these equations cannot be obtained by analytical methods. For this reason, studies of separated flows of this type have stimulated the numerical analysis of the boundary-layer equations, supplemented with interaction conditions (see the review by Ruban (1990)). However, almost all of these methods were developed for non-periodic flows, i.e. with condition (1.4). Examples of steady problems with spatially periodic interaction can be found in Sychev & Sychev (1995), where linearized problems may be solved analytically. A nonlinear problem was solved in Smith & Timoshin (1996), where the flow under consideration was induced by a rotating system of blades. The blade thickness was assumed to be an $O(Re^{-1/2})$ quantity, where Re denotes the Reynolds number, assumed to be large. The numerical solution was calculated by marching over a sequence of angular periods until agreement between the final and initial profiles was achieved. One difficulty in solving problems of this type is that the initial profile is not prescribed but must be found by using the periodicity condition (1.5). In this paper, a numerical method is proposed for solving the steady equations of the interaction theory for flows that are periodic with respect to one coordinate. The method, which extends the analysis presented in Korolev (1987), is characterized by a high rate of convergence (five to six iteration steps are sufficient to reduce the error of a solution to the nonlinear equations to a quantity of the order of 10^{-6}), which does not depend on the mesh size and the spatial extent of the separation region. The method is based on an efficient inversion algorithm for the Jacobi matrix, which makes it possible to implement the numerical method on personal computers on 100×100 grids and to obtain a solution in the required range of parameters in just several minutes. Moreover, the method can be used to find additional solutions when non-uniqueness exists. Here, the method is applied to the problem of the interaction between the boundary layer on the surface of a slightly elliptic cylinder rotating in an incompressible fluid and an inviscid external stream (Sychev & Sychev 1995), when the eccentricity is assumed to be $O(Re^{-1/6})$.

In the orthogonal coordinate system tied to the elliptic surface, the boundary-layer flow is described by equations (1.1)–(1.3), (1.5) with

$$\tau = 2, \qquad a = 2\pi.$$

The interaction condition has the form,

$$A(x) = -C_0^2\left(\frac{\Gamma_1}{2\pi} + \sin^2(x)\right), \tag{1.6}$$

where the parameters C_0 and $\varGamma_1$ are connected with the eccentricity of the ellipse and the flow circulation, respectively. The value of C_0 is prescribed and $\varGamma_1$ must be determined by solving the problem. This form of interaction is the result of matching of the solution for the streamfunction in the boundary layer with the solution for the streamfunction in the external inviscid flow following Sychev & Sychev (1995).

2. Construction of periodic boundary-layer solutions

We introduce new variables defined by the relations,

$$Y = \tau^{-1/3}y, \qquad u = \tau^{2/3}U, \qquad v = \tau^{1/3}V, \qquad P = \tau^{4/3}p, \qquad C_0 = \tau^{1/3}c,$$

and rewrite equations (1.1)–(1.3), (1.5)–(1.6) in terms of the vorticity

$$\omega = \frac{\partial U}{\partial y}.$$

Using the continuity equation and the no-slip condition, we obtain the equation,

$$\int_0^y \omega \, \mathrm{d}y_1 \frac{\partial \omega}{\partial x} + \left(\int_0^y y_1 \frac{\partial \omega}{\partial x} \, \mathrm{d}y_1 - y \int_0^y \frac{\partial \omega}{\partial x} \, \mathrm{d}y_1 \right) \frac{\partial \omega}{\partial y} = \frac{\partial^2 \omega}{\partial y^2}, \tag{2.1}$$

which has to be supplemented with the boundary conditions:

$$\omega \to 1 \quad \text{as} \quad y \to \infty, \tag{2.2}$$

$$\left. \frac{\partial \omega(y,x)}{\partial y} \right|_{y=0} = p'(x). \tag{2.3}$$

The interaction condition (1.6), and the periodicity condition (1.5), for the velocity profile are written as

$$\int_0^\infty (\omega - 1) \, \mathrm{d}y + c^2 \left(\frac{\varGamma_1}{2\pi} + \sin^2(x) \right) = 0, \tag{2.4}$$

$$\omega(y,x) = \omega(y, x + 2\pi). \tag{2.5}$$

Following Wood (1957), we determine the unknown $\varGamma_1$ by invoking the condition that the streamlines are closed, which restricts the external flow configuration. Integrating equation (1.1) from $x = 0$ to $x = 2\pi$ and using the fact that both velocity and pressure profiles are periodic, we obtain

$$\int_0^{2\pi} \frac{\partial}{\partial y} \left(VU - \frac{\partial U}{\partial y} \right) \mathrm{d}x = 0.$$

Integrating this result from 0 to y and changing the order of integration, we use the no-slip condition to obtain

$$\int_0^{2\pi} [V(x,y)U(x,y) - \omega(x,y) + \omega(x,0)] \, \mathrm{d}x = 0. \tag{2.6}$$

From equation (1.1) in the limit of large y, the streamfunction behaves as

$$\varPsi = y^2/2 + A(x)y + B(x) + \cdots,$$

where A and B are periodic functions such that

$$p'(x) = A(x)A'(x) - B'(x).$$

Therefore, condition (2.6) can be rewritten in the following final form,

$$\int_0^{2\pi} [-p'(x)A(x) - 1 + \omega(x,0)]\,\mathrm{d}x = 0, \tag{2.7}$$

and we seek a solution in the range $[0, 2\pi]$. We introduce a non-uniform grid (y_k, x_j), $k = 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$, such that the mesh size along the y-axis is smallest at the surface. Introducing $\Delta y_k = y_k - y_{k-1}$, $\Delta x_j = x_j - x_{j-1}$, and $\omega_{kj} = \omega(y_k, x_j)$ and assuming that the upper boundary of the grid is sufficiently far from the surface, we rewrite condition (2.2) as

$$L_{Mj} = \omega_{Mj} - 1 = 0. \tag{2.8}$$

On the lower boundary of the grid, we write condition (2.4) as follows:

$$L_{1j} = \frac{1}{2}\left[\sum_{k=2}^{M-1} (\omega_{kj} - 1)(y_{k+1} - y_{k-1}) + (\omega_{1j} - 1)\,\mathrm{d}y_1 + (\omega_{Mj} - 1)\,\mathrm{d}y_M\right] + c^2\left(\frac{\Gamma_1}{2\pi} + \sin^2(x_j)\right) = 0. \tag{2.9}$$

Equation (1.1) is approximated by two finite-difference schemes, firstly a second-order accurate Crank–Nicolson scheme in the forward-flow region and a first-order accurate scheme (with respect to Δx) in the reverse-flow region, namely,

$$L_{kj} = (u_{kj} + u_{k,j-\alpha})\frac{\omega_{kj} - \omega_{k,j-\alpha}}{2(x_j - x_{j-\alpha})} + v_{k,j-\alpha/2}(\lambda_y^\beta \omega_{kj} + \lambda_y^\beta \omega_{k,j-\alpha})/2 - (\lambda_{yy}\omega_{kj} + \lambda_{yy}\omega_{k,j-\alpha})/2 = 0, \qquad k = 2, 3, \ldots, M-1, \quad j = 2, 3, \ldots, N,$$

$$u_{kj} = \frac{1}{2}\left[\omega_{1j}\,\mathrm{d}y_1 + \omega_{kj}\,\mathrm{d}y_k + \sum_{l=2}^{M-1} \omega_{lj}(y_{l+1} - y_{l-1})\right],$$

$$v_{k,j-\alpha/2} = [2(x_{j-\alpha} - x_j)]^{-1} \times \left[(\omega_{1j} - \omega_{1,j-\alpha})\,\mathrm{d}y_2\, y_k + \sum_{l=2}^{k-1}(\omega_{lj} - \omega_{l,j-\alpha})(y_k - y_l)(y_{l+1} - y_{l-1})\right],$$

$$\lambda_y^\beta \omega_{kj} = \frac{-\omega_{k-2\beta,j}\xi^2 + \omega_{k-\beta,j} - \omega_{kj}(1 - \xi^2)}{(y_{k-2\beta} - y_k)(\xi - \xi^2)},$$

$$\lambda_{yy}\omega_{kj} = \frac{\nu\omega_{k-2\beta,j} + \delta\omega_{k-\beta,j} + \omega_{k+\beta,j} - \omega_{k,j}(1 + \delta + \nu)}{0.5(\nu + \delta\xi^2 + \gamma^2)(y_{k-2\beta} - y_k)^2},$$

$$\alpha = \operatorname{sgn} u_{kj}, \qquad \beta = \begin{cases} \operatorname{sgn} v_{k,j-\alpha/2}, & k \neq 2, M-1, \\ -1, & k = 2, \\ 1, & k = M-1, \end{cases}$$

where

$$\gamma = \frac{y_{k+\beta} - y_k}{y_{k-2\beta} - y_k}, \qquad \xi = \frac{y_{k-\beta} - y_k}{y_{k-2\beta} - y_k}, \qquad \delta = \frac{\gamma - \gamma^3}{\xi^3 - \xi}, \qquad \nu = \frac{\gamma^3 - \gamma\xi^2}{\xi^2 - 1}.$$

Secondly, a second-order accurate three-point scheme (with respect to x) in the entire flow region was also implemented, namely,

$$L_{kj} = u_{kj}(\lambda_x^\alpha \omega_{kj}) + v_{k,j}(\lambda_y^\beta \omega_{kj}) - \lambda_{yy}\omega_{kj} = 0,$$
$$k = 2, 3, \ldots, M-1; \quad j = 3, 4, \ldots, N, \tag{2.10}$$

$$v_{k,j} = \frac{1}{2}\left[-y_k \Delta y_2 \lambda_x^1 \omega_{1j} + \sum_{l=2}^{k-1} \lambda_x^1 \omega_{lj}(y_l - y_k)(y_{l+1} - y_{l-1})\right],$$

$$\lambda_x^\alpha \omega_{kj} = (x_j - x_{j-\alpha})^{-1}\left[-\frac{-\omega_{k,j-2\alpha}}{\mu(1-\mu)} + \frac{\omega_{k,j-\alpha}\mu}{1-\mu} + \frac{\omega_{kj}(1+\mu)}{\mu}\right],$$

where

$$\alpha = \operatorname{sgn} u_{kj}, \qquad \beta = \operatorname{sgn} v_{kj}, \qquad \mu = \frac{x_j - x_{j-2\alpha}}{x_j - x_{j-\alpha}}.$$

Here, the first-order derivatives with respect to x and y are approximated by finite differences depending on the signs of the streamwise and normal velocity components, respectively (Carter 1974; Ruban 1978). The periodicity conditions for the vorticity profile and condition (2.3) are written as

$$Q_k = \omega(k, 1) - \omega(k, N) = 0, \qquad k = 1, 2, \ldots, M, \tag{2.11}$$

$$p_j' = \lambda_y^{-1}\omega_{1j}. \tag{2.12}$$

It should be noted, however, that the equations for the vorticity can be solved by a condition obtained by integrating the boundary condition (2.3) from 0 to 2π, which is simpler than expression (2.7), namely,

$$p(x + 2\pi) - p(x) = \int_x^{x+2\pi} \left.\frac{\omega(y, z)}{y}\right|_{y=0} \mathrm{d}z = 0. \tag{2.13}$$

In finite-difference form, this is written as

$$Q_{M+1} = \sum_{j=2}^{N} (\lambda_y^{-1})(\omega_{1j} + \omega_{1,j-1})\Delta x_j / 2 = 0. \tag{2.14}$$

When a set of values of Γ_1 is used instead of (2.7) for a particular value of c, the numerical solution (when obtainable) cannot be matched with the inviscid form along the upper grid boundary and the resulting pressure distribution is not periodic. This set of equations should be supplemented with an additional equation

$$Q_{M+2} = c - c_0 = 0,$$

where c_0 is a prescribed value of the parameter. However, since Γ_1 is characterized by a singular behaviour near any values of c beyond which solutions do not exist locally and prior to which two solutions may exist locally, the arclength s relating

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the value of Γ_1 (which depends on the interaction region) to c should be used as a parameter:

$$\mathrm{d}s = \sqrt{\mathrm{d}^2\Gamma_1 + \mathrm{d}^2 c}. \tag{2.15}$$

The parameter s becomes a prescribed quantity, and c and Γ_1 are treated as unknown variables. This modification is characteristic of the so-called continuation method (Keller 1977). In finite-difference form, the modified equation is written as

$$Q_{M+2} = \frac{(c - c^a)(c^a - c^b) + (\Gamma_1 - \Gamma_1^a)(\Gamma_1^a - \Gamma_1^b)}{((c^a - c^b)^2 + (\Gamma_1^a - \Gamma_1^b)^2)^{1/2}} - \mathrm{d}s = 0, \tag{2.16}$$

where c^a, Γ_1^a , c^b, Γ_1^b are the values of c and Γ_1 previously obtained at adjacent points a and b on the curve representing the solution. One advantage of this approach is that the solution can be continued beyond the singular points to other solution branches (when they exist). This equation is used in the analysis that follows. The problem for ω posed here can be solved without calculating pressure p explicitly. Then, the pressure distribution can be found as an integral of the pressure gradient given by (2.12).

The set of nonlinear algebraic equations derived above is solved by applying the Newton–Kantorovich method and Gaussian elimination to invert the Jacobi matrix at each iteration step. We introduce vectors $\boldsymbol{W}_j$ and $\boldsymbol{P}$, with dimensions equal to M and $N_1 = M + 2$, respectively, as follows:

$$\boldsymbol{W}_j = [\omega_{1j}, \omega_{2j}, \dots, \omega_{Mj}]^{\mathrm{T}}, \qquad \boldsymbol{P} = [\omega_{11}, \omega_{21}, \dots, \omega_{M1}, \Gamma_1, c]^{\mathrm{T}}.$$

Henceforth, we assume that the boundary layer is unseparated in the interval (x_1, x_2). Otherwise, we can perform a preliminary analysis of the interaction condition and determine the region where the flow accelerates, or carry out a series of preliminary computations using the periodicity condition, to set the grid origin, $j = 1$, in such a flow region. Then, the set of equations can be written in compact form as

$$\boldsymbol{L}_2(\boldsymbol{W}_1, \boldsymbol{W}_2, \boldsymbol{P}) = \boldsymbol{0}, \tag{2.17}$$

$$\begin{gathered}\boldsymbol{L}_j(\boldsymbol{W}_{j-2}, \boldsymbol{W}_{j-1}, \boldsymbol{W}_j, \boldsymbol{W}_{j+1}, \boldsymbol{W}_{j+2}, \boldsymbol{P}) = \boldsymbol{0}, \\ \boldsymbol{L}_j = [L_{1j}, L_{2j}, \dots, L_{kj}, \dots, L_{Mj}]^{\mathrm{T}}, \qquad j = 3, 4, \dots, N,\end{gathered}$$

$$\boldsymbol{Q}(\boldsymbol{W}_1, \boldsymbol{W}_2, \dots, \boldsymbol{W}_N, \boldsymbol{P}) = [Q_1, \dots, Q_{N_1}]^{\mathrm{T}} = 0, \tag{2.18}$$

where $L_{k,j} = 0$ is the corresponding finite-difference equation at the kth point on the jth vertical grid line. Suppose $\boldsymbol{W}_j^i$ and $\boldsymbol{P}^i$ are obtained as approximations of the corresponding vectors at the ith iteration step. To improve the approximation of these vectors, we introduce the corresponding corrections, $\Delta\boldsymbol{W}_j$ and $\Delta\boldsymbol{P}$, and rewrite equations (2.17) as

$$\begin{aligned}&\frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{W}_{j+2}}\Delta\boldsymbol{W}_{j+2} + \frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{W}_{j+1}}\Delta\boldsymbol{W}_{j+1} + \frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{W}_{j}}\Delta\boldsymbol{W}_{j} + \frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{W}_{j-1}}\Delta\boldsymbol{W}_{j-1} \\ &\quad + \frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{W}_{j-2}}\Delta\boldsymbol{W}_{j-2} + \frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{P}}\Delta\boldsymbol{P} + \boldsymbol{L}_j(\boldsymbol{W}_{j-2}^i, \boldsymbol{W}_{j-1}^i, \boldsymbol{W}_j^i, \boldsymbol{W}_{j+1}^i, \boldsymbol{W}_{j+2}^i, \boldsymbol{P}^i) = \boldsymbol{0}, \\ &\qquad j = 3, 4, \dots, N,\end{aligned} \tag{2.19}$$

$$\frac{\partial \boldsymbol{L}_2}{\partial \boldsymbol{W}_1}\Delta \boldsymbol{W}_1 + \frac{\partial \boldsymbol{L}_2}{\partial \boldsymbol{W}_2}\Delta \boldsymbol{W}_2 + \frac{\partial \boldsymbol{L}_2}{\partial \boldsymbol{P}}\Delta \boldsymbol{P} + \boldsymbol{L}_2(\boldsymbol{W}_2^i, \boldsymbol{W}_1^i, \boldsymbol{P}^i) = \boldsymbol{0}, \tag{2.20}$$

where the matrices multiplied by the correction vectors are calculated as

$$\left.\begin{aligned}\frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{W}_r} &= \left\{\frac{\partial L_{kj}}{\partial W_{ms}}\right\},\\ \frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{P}} &= \left\{\frac{\partial L_{kj}}{\partial P_l}\right\},\end{aligned}\right\} \quad \begin{aligned} k = 1, 2, \ldots, M, \quad l = 1, \ldots, N_1 \quad m = 1, 2, \ldots, M,\\ r = j-2, \ldots, j+2 \quad s = 1, 2, \ldots, M.\end{aligned}$$

Here, P_l is the lth component of the vector $\boldsymbol{P}$. We seek a solution to the problem (2.19), (2.20) in the form,

$$\Delta \boldsymbol{W}_j = R_j \Delta \boldsymbol{W}_{j+1} + T_j \Delta \boldsymbol{W}_{j+2} + Z_j \Delta \boldsymbol{P} + \boldsymbol{S}_j, \tag{2.21}$$

where $\boldsymbol{R}_j$ and $\boldsymbol{T}_j$ are $M \times M$ square matrices, $\boldsymbol{Z}_j$ is an $M \times N$ matrix, and $\boldsymbol{S}_j$ is an M-component vector. Using (2.21), we solve equations (2.19) and (2.20) for $3 \leqslant j \leqslant N$ to find (as in the conventional Gaussian elimination for five-point equations) recursive relations to be used in calculating the matrices $\boldsymbol{R}_j$, $\boldsymbol{T}_j$, $\boldsymbol{Z}_j$, and $\boldsymbol{S}_j$:

$$\begin{aligned}\boldsymbol{R}_j &= \boldsymbol{C}\left(\boldsymbol{B}\boldsymbol{T}_{j-2} + \frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{W}_{j+1}}\right),\\ \boldsymbol{T}_j &= \boldsymbol{C}\frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{W}_{j+2}},\\ \boldsymbol{Z}_j &= \boldsymbol{C}\left(\boldsymbol{B}\boldsymbol{Z}_{j-1} + \frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{W}_{j-2}}\boldsymbol{Z}_{j-2} + \frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{P}}\right),\\ \boldsymbol{S}_j &= \boldsymbol{C}\left(\boldsymbol{B}\boldsymbol{S}_{j-1} + \frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{W}_{j-2}}\boldsymbol{S}_{j-2} + \boldsymbol{L}_j\right),\end{aligned}$$

where

$$\boldsymbol{C} = -\left(\boldsymbol{B}\boldsymbol{R}_{j-1} + \frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{W}_{j-2}}\boldsymbol{T}_{j-2} + \frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{W}_j}\right)^{-1}, \qquad \boldsymbol{B} = \frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{W}_{j-2}}\boldsymbol{R}_{j-2} + \frac{\partial \boldsymbol{L}_j}{\partial \boldsymbol{W}_{j-1}}.$$

Invoking the definition of the vector $\boldsymbol{P}$ and equation (2.20), we find the initial values:

$$\boldsymbol{R}_1 = \boldsymbol{T}_1 = \boldsymbol{S}_1 = \boldsymbol{0}, \qquad \boldsymbol{Z}_1 = \begin{cases} Z_{kk} = 1, \ k = 1, \ldots, M, \\ Z_{k,l} = 0, \ l = 1, \ldots, N_1, \ l \neq k, \end{cases} \qquad \boldsymbol{R}_2 = \boldsymbol{T}_2 = \boldsymbol{0},$$

$$\boldsymbol{Z}_2 = -\left(\frac{\partial \boldsymbol{L}_2}{\partial \boldsymbol{W}_2}\right)^{-1}\frac{\partial \boldsymbol{L}_2}{\partial \boldsymbol{P}}, \qquad \boldsymbol{S}_2 = -\left(\frac{\partial \boldsymbol{L}_2}{\partial \boldsymbol{W}_2}\right)^{-1}\left(\boldsymbol{L}_2 + \frac{\partial \boldsymbol{L}_2}{\partial \boldsymbol{W}_1}\right),$$

which are required to start calculations based on the recursive relations. We assume that the region of unseparated flow also includes the interval (x_{N-1}, x_N) at the right boundary of the grid. Then, $\boldsymbol{R}_{N-1} = \boldsymbol{R}_N = \boldsymbol{T}_{N-1} = \boldsymbol{T}_N = \boldsymbol{0}$ and therefore we have the following:

$$\Delta \boldsymbol{W}_{N-1} = \boldsymbol{Z}_{N-1}\Delta \boldsymbol{P} + \boldsymbol{S}_{N-1}, \qquad \Delta \boldsymbol{W}_N = \boldsymbol{Z}_N \Delta \boldsymbol{P} + \boldsymbol{S}_N.$$

Using these equations, we write equation (2.21) in a simpler form, namely,

$$\Delta \boldsymbol{W}_j = \boldsymbol{Z}'_j \Delta \boldsymbol{P} + \boldsymbol{S}'_j, \qquad j = 3, 4, \ldots, N, \tag{2.22}$$

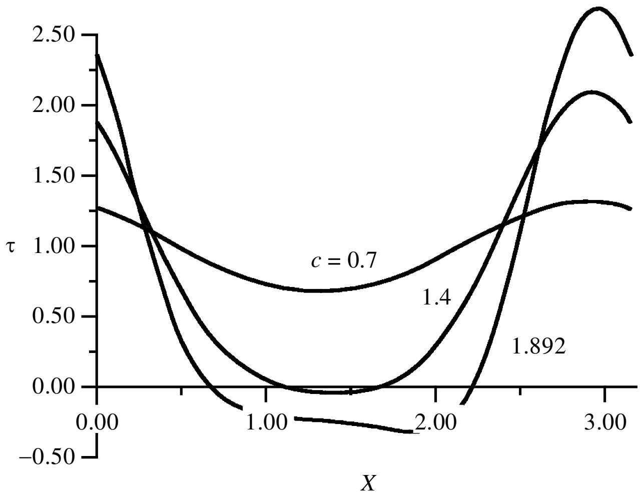

Figure 1. The skin friction distribution on the body surface for various values of c.

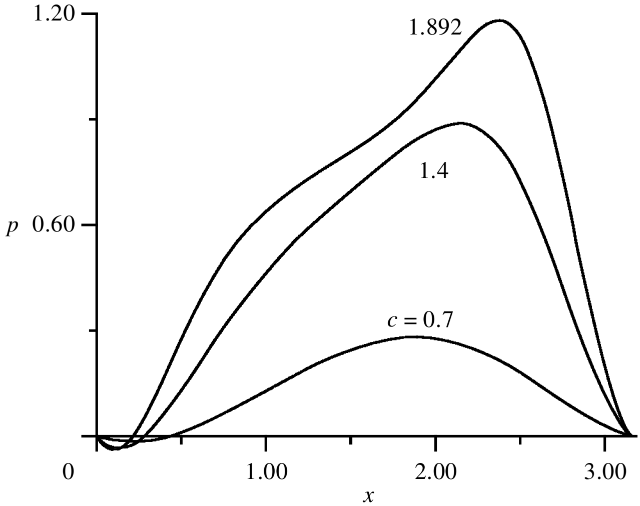

Figure 2. The pressure distribution on the body surface for various values of c.

where $\boldsymbol{Z}'_j$ and $\boldsymbol{S}'_j$ are determined by the recursive relations,

$$\boldsymbol{Z}'_j = \boldsymbol{Z}_j + \boldsymbol{R}_j\boldsymbol{Z}'_{j+1} + \boldsymbol{T}_j\boldsymbol{Z}'_{j+2}, \qquad \boldsymbol{S}'_j = \boldsymbol{S}_j + \boldsymbol{R}_j\boldsymbol{S}'_{j+1} + \boldsymbol{T}_j\boldsymbol{S}'_{j+2},$$

with the initial values

$$\boldsymbol{Z}'_N = \boldsymbol{Z}_N, \qquad \boldsymbol{S}'_N = \boldsymbol{S}_N, \qquad \boldsymbol{Z}'_{N-1} = \boldsymbol{Z}_{N-1}, \qquad \boldsymbol{S}'_{N-1} = \boldsymbol{S}_{N-1}.$$

To determine the unknown vector $\Delta\boldsymbol{P}$, we use conditions (2.11), (2.13) and (2.16), writing them in the form,

$$\boldsymbol{Q}(\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots, \boldsymbol{W}_N, \boldsymbol{P}) = [Q_1, \ldots, Q_{N_1}]^{\mathrm{T}} = 0.$$

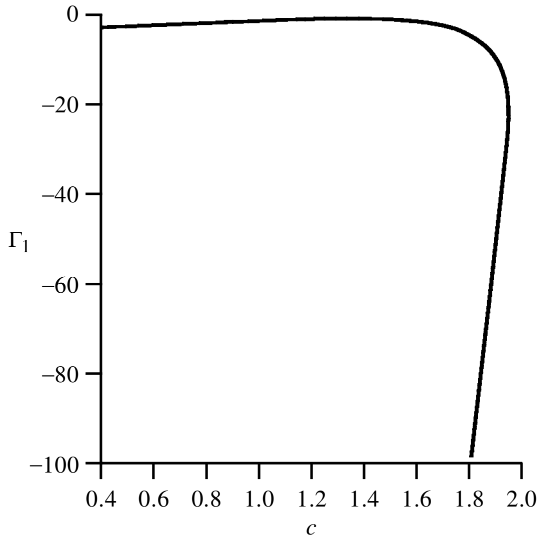

Figure 3. Dependence of the value Γ_1 on the value of c.

This set of equations yields

$$\sum_{j=1}^{N} \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{W}_j} \Delta \boldsymbol{W}_j + \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{P}} \Delta \boldsymbol{P} + \boldsymbol{Q}(\boldsymbol{W}_1^i, \boldsymbol{W}_2^i, \ldots, \boldsymbol{W}_N^i, \boldsymbol{P}^i) = \boldsymbol{0}, \tag{2.23}$$

where the matrices $\boldsymbol{Q}_W$ and $\boldsymbol{Q}_P$ are calculated as

$$\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{W}_j} = \left\{ \frac{\partial Q_r}{\partial W_{kj}} \right\}, \qquad \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{P}} = \left\{ \frac{\partial Q_r}{\partial P_l} \right\},$$
$$r = 1, 2, \ldots, N_1, \quad k = 1, 2, \ldots, M, \quad l = 1, 2, \ldots, N_1.$$

Substituting $\Delta \boldsymbol{W}_j$ given by equation (2.22) into equation (2.23), we obtain an equation for $\Delta \boldsymbol{P}$, namely,

$$\left(\sum_{j=1}^{N} \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{W}_j} \boldsymbol{Z}_j' + \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{P}} \right) \Delta \boldsymbol{P} + \sum_{j=1}^{N} \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{W}_j} \boldsymbol{S}_j' + \boldsymbol{Q} = \boldsymbol{0}.$$

Solving this equation for $\Delta \boldsymbol{P}$, we find all values of $\Delta \boldsymbol{W}_j$ from equation (2.22) and the next approximation for $\Delta \boldsymbol{W}_j$ and $\Delta \boldsymbol{P}$ is sought in the form,

$$\Delta \boldsymbol{W}_j^{i+1} = \boldsymbol{W}_j^i + \Delta \boldsymbol{W}_j, \qquad \boldsymbol{P}^{i+1} = \boldsymbol{P}^i + \Delta \boldsymbol{P}.$$

It should be noted that the calculations of $\boldsymbol{R}_j$ and $\boldsymbol{T}_j$, as well as the recalculation of $\boldsymbol{Z}_j$ and $\boldsymbol{S}_j$, are necessary only for the reverse-flow region.

3. Results

The proposed method was used to compute the flow on an $N \times M = 201 \times 201$ grid with non-uniform mesh size along the y-axis and a uniform mesh size along

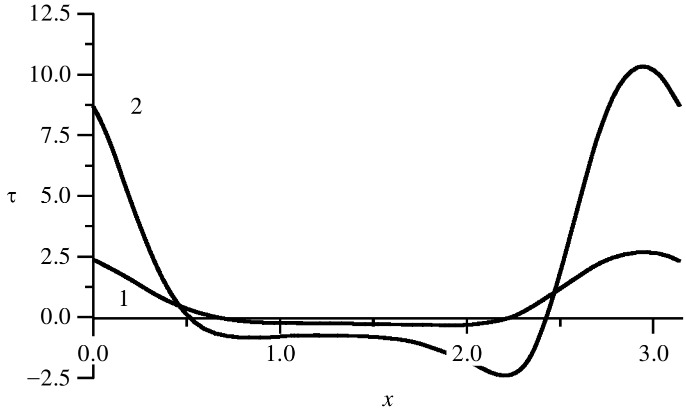

Figure 4. Friction distribution on the body surface for $c = 1.82$ but different values of Γ_1.

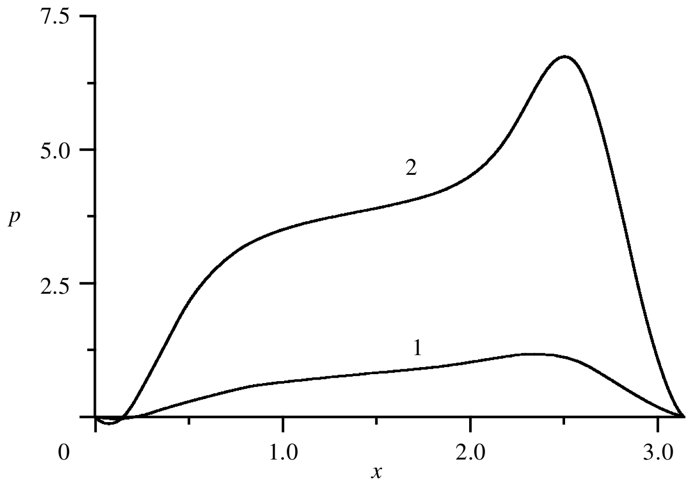

Figure 5. Pressure distribution on the body surface for $c = 1.82$ but different values of Γ_1.

the x-axis. The mesh size was the smallest ($\Delta y = 0.1$) near the surface irregularity. Originally, the computations were performed within $-\pi \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant 25$, but since the elliptical cross-section is symmetrical, the analysis was subsequently restricted to $0 \leqslant x \leqslant \pi$. However, identical numerical results were obtained on both grids. The set of equations (2.1)–(2.7) was first solved for $c = 0$, and then solutions were obtained for gradually increasing values of c. For $c < 1.6$, the parameter c was held constant during each iterative cycle. The value of c was then increased, and the computations were repeated. When the value of c was large, solutions were characterized by a singular behaviour. The small increase of c reduced to the huge change of Γ_1. For this reason, the arclength s, as defined by equation (2.15), was used as a parameter in the problem instead of c. In accordance with the continuation method, the value of s was held constant, and c and Γ_1 were treated as unknown quantities. Solutions obtained for values of s less than 0.1 were used as starting approximations for the flow field. The three-point scheme in (2.10) was found to be the most efficient. Even though its computational cost was higher than that of the Crank–Nicolson scheme, in terms of the number of arithmetic operations per

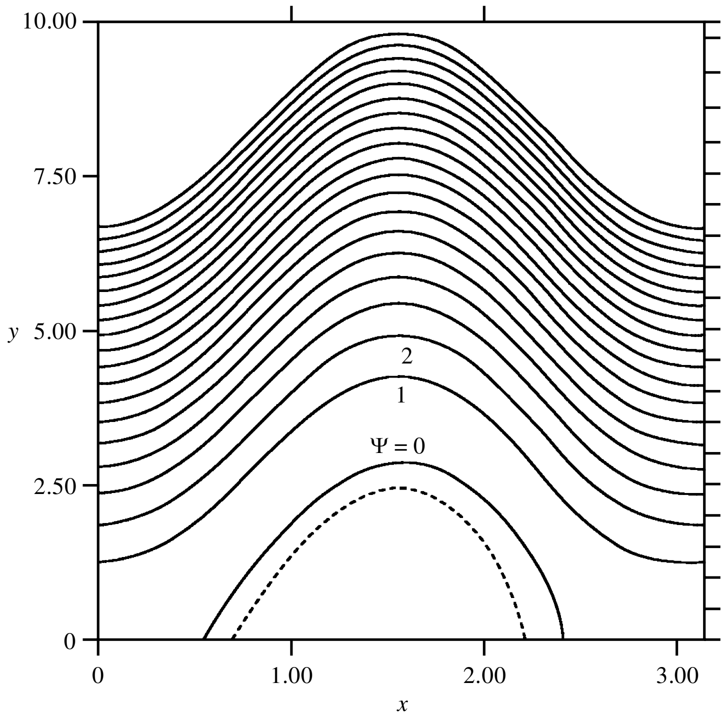

Figure 6. Streamline pattern for $c = 1.82$.

iteration, it proved to be less sensitive to the starting approximation for the flow field and was characterized by a higher convergence rate of the iterative process. Five to six iteration steps were required for a solution to converge with a residual of the order of 10^{-6}, namely,

$$\max |\Delta\omega_{kj}|^i < 10^{-6},$$

whereas the Crank–Nicolson scheme required one or two additional iteration steps to achieve a similar accuracy. Grid-size convergence was checked by reducing the mesh by half. The largest discrepancy in the values of the friction (within 4%) was observed in the vicinity of the pressure peak. The results computed with these schemes were very close (the maximum difference in the skin friction distribution is of the order of 10^{-3}). A numerical analysis also showed that the difference between solutions obtained under conditions (2.7) and (2.13) was insignificant (the maximum difference in pressure distribution and in the skin friction distribution is on the order of 10^{-3}). When one of the conditions was met, the other was also satisfied. Here, the results obtained with the use of the three-point scheme (with respect to x) are described. Figures 1 and 2 show the friction and, respectively, the pressure distributions on the surface of an elliptic cylinder for $c = 0.7$, 1.4, and 1.892. According to figure 1, a reverse-flow region forms at $c = 1.4$ and monotonically expands as the parameter c increases. One would expect a solution to exist for all values of c. However, this is not the case with the periodic boundary-layer analysed in this study. Figure 3 shows the circulation Γ_1 as a function of c and it shows that when c exceeds $c^* = 1.95 \pm 0.05$, the problem has no solution. Another characteristic feature of the solution is the existence of a second lower branch, which is characterized by decrease in Γ_1 and an increase in the extent of the reverse-flow region associated with a decrease in c. It should be noted that a non-unique solution was also obtained in the theory of interaction between boundary layers and external streams in earlier studies of laminar incompressible flow past the leading edge of a slender airfoil in Brown & Stewartson (1983) and Ruban (1982), jet flows past a curved surface in Zametaev (1986) and the trailing edge of a slender airfoil in Korolev (1989), incompressible flow past the vertex of an obtuse angle in work of Korolev (1991), and supersonic viscous flow past an axially symmetric surface in Gittler & Kluwick (1987). Figures 4 and 5 compare the skin friction and the pressure distributions on the surface of the elliptic cylinder obtained for $c = 1.82$ and two different values of Γ_1 and the curves 1 and 2 represent the upper and, respectively, lower branches of the solution. The flow pattern computed for the lower branch at $c = 1.82$ is shown in figure 6. Here streamlines are shown in the regime $0 \leqslant \psi \leqslant 20$ with the constant interval $\Delta\psi = 1$. The broken curve represents the isoline of zero streamfunction in the flow pattern computed for the upper branch of the solution. However, the reverse-flow region expands and the boundary-layer displacement thickness increases when the computations were continued further along the lower branch. This resulted in a loss of accuracy, which made it impossible to perform a numerical analysis of the lower branch for values of Γ_1 lower than those represented in figure 3.

In summary, the present analysis shows that the solution exists for a limited range of c. This may indicate that the initial assumptions about the time-independent and/or unseparated regime of the external flow are not valid when $c > c^*$, and a time-dependent solution should instead be sought.

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References

Brown, S. N. & Stewartson, K. 1983 On integral equation of marginal separation. *SIAM J. Appl. Math.* **43**, 1119–1126.

Carter, J. E. 1974 Solutions for laminar boundary layers with separation and reattachment. AIAA Paper. no. 74-583.

Gittler, P. & Kluwick, A. 1987 Triple deck solutions for supersonic flows past flared cylinders. *J. Fluid Mech.* **179**, 469–487.

Keller, H. B. 1977 In *Applications of bifurcation theory* (ed. P. Rabinowitz), pp. 359. Academic Press.

Korolev, G. L. 1987 A method of solving problems of the asymptotic theory between boundary layer and external flow. *Zh. Vychisl. Mat. Mat. Fiz.* **27**, 1224–1232. (English transl. *USSR Comput. Math. Math. Phys.* **27**, 175–181.)

Korolev, G. L. 1989 Theory of separated flow over the trailing edge of a thin profile. *Mekh. Zhidk. Gaza.* **4**, 55–59. (English transl. *Fluid Dyn.* **24**, 534–537.)

Korolev, G. L. 1991 Interaction theory and non-uniqueness of separated flows around solid bodies. In *Separated flows and jets* (ed. V. V. Kozlov & A. V. Dovgal), pp. 139–142. Springer.

Ruban, A. I. 1978 Numerical solution of the local asymptotic problem of the unsteady separation of a laminar boundary layer in a supersonic flow. *Zh. Vychisl. Mat. Mat. Fiz.* **18**, 1263–1265.

Ruban, A. I. 1982 Asymptotic theory of short separation regions at the leading edge of a thin airfoil. *Mech. Zhid. Gaza* **1**, 23–28.

Ruban, A. I. 1990 Numerical methods in the theory of boundary layer–inviscid flow interaction. *Uch. Zap. TsAGI.* **21**, 3–25.

Smith, F. T. & Timoshin, S. N. 1996 Blade–wake interactions and rotary boundary layers. *Proc. R. Soc. Lond.* A **452**, 1301–1329.

Sychev, V. V. & Sychev Jr, V. V. 1995 Viscous flow a rotating elliptic cylinder *Comp. Math. Phys.* **35**, 799–804. (English transl. *Zh. Vychisl. Mat. Mat. Fiz.* **35**(6), 1001–1008.)

Sychev, V. V., Ruban, A. I., Sychev, Vic. V. & Korolev, G. L. 1998 *Asymptotic theory of flow separation.* Cambridge University Press.

Wood, W. W. 1957 Boundary layers whose streamlines are closed. *J. Fluid Mech.* **2**, 77–87.

Zametaev, V. B. 1986 Existence and nonuniqueness of local separation zones in viscous jets. *Mech. Zhid. Gaza* **1**, 38–45. (English transl. *Fluid Dyn.* **21**, 31–38.)

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